The Maximum Eccentricity Energy of a Graph

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Abstract— In This paper, we introduce the concept of a maximum eccentricity matrix $M_e(G)$ of a connected graph G and obtain some coefficients of the characteristic polynomial $P(G,\lambda)$ of the maximum eccentricity matrix of G. We also introduce the maximum eccentricity energy $EM_e(G)$ of a connected graph. Maximum eccentricity energies of some well-known graphs are obtained. Upper and lower bounds for $EM_e(G)$ are established. It is shown that if G is a self-centered k-regular graph with diameter, then kD is a maximum eccentricity eigenvalue of G and $EM_e(G)=DE(G)$. Moreover, it is also shown that if the maximum eccentricity energy of a graph is rational then it must be an even.

Index Terms: *Distance in graphs, maximum eccentricity matrix, maximum eccentricity eigenvalues, maximum eccentricity energy of a graph..*

1. Introduction

In this paper, all graphs are assumed to be simple, finite and connected. A graph $G=(V,E)$ is a simple graph, that is, having no loops, no multiple and directed edges. As usual, we denote by $n=|V|$ and $m=|E|$ to the number of vertices and edges in a graph G, respectively. For a vertex $v\in V$, the open neighborhood of v in a graph G, denoted $N(v)$, is the set of all vertices that are adjacent to v and the closed neighborhood of v is $N[v]=N(v)\cup\{v\}$. The degree of a vertex v in G is $d(v)=|N(v)|$. A graph G is said to be k-regular graph if $d(v)=k$ for every $v\in V(G)$. The distance $d(u,v)$ between any two vertices u and v in a graph G is the length of a minimum path connecting them. For a vertex v of G, the eccentricity of a vertex v is $e(v)=\max\{d(v,u):u\in V(G)\}$. The radius of G is $r(G)=\min\{e(v):v\in V(G)\}$ and the diameter of G is $D(G)=\max\{e(v):v\in V(G)\}$.Hence, $r(G)\le e(v)\le D(G)$, for every $v\in V(G)$. A vertex v in a connected graph G is a central vertex if $e(v)=r(G)$, while a vertex v in a connected graph G is a peripheral vertex if $e(v)=D(G)$. A graph G is called a self-centered graph if $e(v)=r(G)=D(G)$ for every vertex $v\in V(G)$ [2]. We denote K_n, C_n, $K_{1,n-1}$ and $K_{r,s}$ the complete, cycle, star and complete bipartite graph respectively. For graph notation and terminology we refer to books [5, 8].

The concept energy of a graph introduced by I. Gutman [6], in the year 1978. Let G be a graph with n vertices and m edges and let $A(G)=(a_{ij})$ be the adjacency matrix of G, where

$$a_{ij}=\begin{cases}1, & if\ v_iv_j\in E(G);\\ 0, & othrwise.\end{cases}$$

The eigenvalues $\lambda_1,\lambda_2,\ldots,\lambda_n$ of a matrix A(G), assumed in non-increasing order, are the eigenvalues of the graph G. Let $\lambda_1\ge\lambda_2\ge\ldots\ge\lambda_t$, for $t\le n$ be the distinct eigenvalues of G with multiplicity $m_1,m_2,\ldots,m_t$, respectively, the multiset of eigenvalues of $A(G)$ is called the spectrum of G and denoted by

$$Sp(G)=\begin{pmatrix}\lambda_1 & \lambda_2 & \ldots & \lambda_t\\ m_1 & m_2 & \ldots & m_t\end{pmatrix}.$$

As A is real symmetric with zero trace, the eigenvalues of G are real with sum equal to zero. The energy $E(G)$ of a graph G is defined to be the sum of the absolute values of the eigenvalues of G, i.e.,

$$E(G)=\sum_{i=1}^{n}|\lambda_i|.$$

For more details on the mathematical aspects of the theory of graph energy we refer to [3, 7, 12] and the references therein.

Prof. C. Adiga and M. Smitha [1], have defined the maximum degree energy $E_M(G)$ of a graph G which depend on the maximum degree matrix $M(G)$ of G. Let G be a simple graph with n vertices $v_1,v_2,\ldots,v_n$. Then the maximum degree matrix $M(G)$ of a graph G defines as, where

$$d_{ij} = \begin{cases} \max\{d(v_i), d(v_j)\}, & If\ v_i v_j \in E(G); \\ 0, & otherwise. \end{cases}$$

As $M(G)$ is real symmetric with zero trace, then the eigenvalues of G being real with sum equal to zero. Motivated by this paper, we introduce the concept of a maximum eccentricity matrix $M_e(G)$ of a connected graph G and obtain some coefficients of the characteristic polynomial $P(G,\lambda)$ of the maximum eccentricity matrix of G. We also introduce the maximum eccentricity energy $EM_e(G)$ of a connected graph G. Maximum eccentricity energies of some well-known graphs are obtained. Upper and lower bounds for $EM_e(G)$ are established. It is shown that if G is a self-centered k-regular graph with diameter D, then kD is a maximum eccentricity eigenvalue of G and $EM_e(G) = DE(G)$. Moreover, it is also shown that if the maximum eccentricity energy of a graph is rational then it must be an even integer.

2. THE MAXIMUM ECCENTRICITY ENERGY OF GRAPHS

Let $G(V,E)$ be a simple connected graph with n vertices $v_1, v_2, ..., v_n$ and let $e(v_i)$ be the eccentricity of a vertex v_i, $i = 1,2,...,n$. The maximum eccentricity matrix of G defining as, where

$$e_{ij} = \begin{cases} \max\{e(v_i), e(v_j)\}, & If\ v_i v_j \in E(G); \\ 0, & otherwise. \end{cases}$$

The characteristic polynomial of the maximum eccentricity matrix $M_e(G)$ is defined by

$$P(G,\lambda) = det(\lambda I - M_e(G)).$$

where I is the unit matrix of order n. The maximum eccentricity eigenvalues of G are the eigenvalues of $M_e(G)$. Since $M_e(G)$ is real and symmetric with zero trace, then its eigenvalues are real numbers with sum equals to zero. We label them in non-increasing order $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$. The maximum eccentricity energy of a graph G is defined as

$$EM_e(G) = \sum_{i=1}^{n} |\lambda_i|.$$

To illustrate this concept, we study the following examples.

Example 2.1 Let G_1 be a graph in Figure 1.

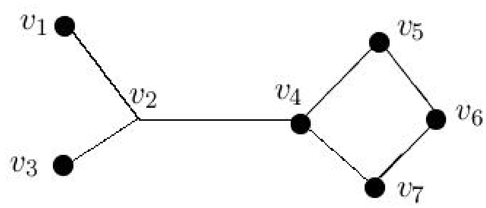

Figure 1: G_1

Then the maximum eccentricity matrix of G_1 is

$$M_e(G_1) = \begin{pmatrix} 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 4 & 3 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 3 & 0 & 3 \\ 0 & 0 & 0 & 3 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 4 \\ 0 & 0 & 0 & 3 & 0 & 4 & 0 \end{pmatrix}_{(7\times 7)}$$

The characteristic polynomial of $M_e(G_1)$ is

$$P(G_1,\lambda) = det(\lambda I - M_e(G))$$

$$= \begin{vmatrix} \lambda & -4 & 0 & 0 & 0 & 0 & 0 \\ -4 & \lambda & -4 & -3 & 0 & 0 & 0 \\ 0 & -4 & \lambda & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & \lambda & -3 & 0 & -3 \\ 0 & 0 & 0 & -3 & \lambda & -4 & 0 \\ 0 & 0 & 0 & 0 & -4 & \lambda & -4 \\ 0 & 0 & 0 & -3 & 0 & -4 & \lambda \end{vmatrix}_{(7\times 7)}$$

$$= \lambda^7 - 91\lambda^5 + 1888\lambda^3.$$

$$= \lambda^3(\lambda^2 - 32)(\lambda^2 - 59).$$

Then the maximum eccentricity eigenvalues of G_1 are $\lambda_1 = \sqrt{59}$, $\lambda_2 = 4\sqrt{2}$, $\lambda_3 = \lambda_4 = \lambda_5 = 0$, $\lambda_6 = -4\sqrt{2}$, $\lambda_7 = -\sqrt{59}$.

Therefore, the maximum eccentricity energy of G_1 is

$$EM_e(G_1) = 2\sqrt{59} + 8\sqrt{2}.$$

Example 2. Let G_2 be the graph in Figure 2.

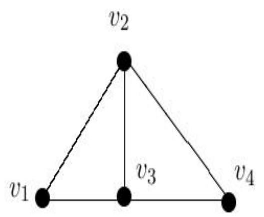

Figure 2: G_2

Then the maximum eccentricity matrix of G_2 is

$$M_e(G_2) = \begin{pmatrix} 0 & 2 & 2 & 0 \\ 2 & 0 & 1 & 0 \\ 2 & 1 & 0 & 2 \\ 0 & 2 & 2 & 0 \end{pmatrix}_{(4\times4)}$$

The characteristic polynomial of $M_e(G_2)$ is

$$P(G_2,\lambda) = \begin{vmatrix} \lambda & -2 & -2 & 0 \\ -2 & \lambda & -1 & 0 \\ -2 & -1 & \lambda & -2 \\ 0 & -2 & -2 & \lambda \end{vmatrix}_{(4\times4)}$$
$$= \lambda^4 - 17\lambda^2 - 16\lambda$$
$$= \lambda(\lambda+1)(\lambda^2 - \lambda - 16)$$

The maximum eccentricity eigenvalues of G_2 are $\lambda_1 = \frac{1+\sqrt{65}}{2},\ \lambda_2 = 0,\ \lambda_3 = -1,\ \lambda_4 = \frac{1-\sqrt{65}}{2}$. Therefore, the maximum eccentricity energy of G_2 is $EM_e(G) = 1+\sqrt{65}.$

3. Properties of Maximum Eccentricity Energy

In this section, we obtain the values of some coefficients of the characteristic polynomial of the maximum eccentricity matrix and investigate some properties of maximum eccentricity eigenvalues of a graph G.

Theorem 3.1 *Let G be a graph of order n and let* $P(G,\lambda) = c_0\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \ldots + c_n$ be the characteristic polynomial of the maximum eccentricity matrix of G. Then

1. $c_0 = 1.$
2. $c_1 = 0.$
3. $c_2 = -\sum_{i=1}^{n}(x_i + y_i)e^2(v_i)$, where
$x_i = |\{u \in N(v_i) : e(u) < e(v_i), 1 \le i \le n\}|$
and
$y_i = |\{u_j \in N(v_i), j > i : e(u_j) = e(v_i), 1 \le i \le n\}|.$
4. $c_3 = -2\sum_{\substack{\Delta v_i v_j v_k \\ e(v_i)\le e(v_j)\le e(v_k)}} e^2(v_k)e(v_j)$.
5. $c_n = det(M_e(G))$.

Proof. The proof is similar in spirit to the proof of Theorem 2.1 in [1].

Remark 3.2 *In Theorem 3.1*

The sum $\sum_{i=0}^{n}(x_i + y_i)$ is the number of edges in the graph G.

Number of terms in the sum $\sum_{\substack{\Delta v_i v_j v_k \\ e(v_i)\le e(v_j)\le e(v_k)}} e^2(v_k)e(v_j)$ is equal to the number of triangles in the graph G. Hence, If the graph G is free-triangle graph, then $c_3 = 0$.

If $M_e(G)$ is singular (has zero eigenvalue), then $c_n = 0$.

Example 3.3 For the graph G_1 in Figure 1, the coefficients c_2 of λ^5 and c_3 of λ^4 in $P(G_1,\lambda)$ are

$$c_2 = -\sum_{i=1}^{n}(x_i + y_i)e^2(v_i)$$
$$= -\begin{bmatrix}(1+0)4^2 + (1+0)3^2 + (1+0)4^2 + (0+0)2^2 + (1+0)3^2 + \\ (2+0)4^2 + (1+0)3^2\end{bmatrix}$$
$$= -[16+9+16+0+9+32+9]$$
$$= -91.$$

Since there is no triangle in G_1, then $c_3 = 0$.

For the graph G_2 in Figure 2, the coefficients c_2 of λ^2 and c_3 of λ are

$$c_2 = -\sum_{i=1}^{n}(x_i + y_i)e^2(v_i)$$
$$= -[(2+0)2^2 + (0+1)1^2 + (0+0)1^2 + (2+0)2^2]$$
$$= -[8+1+0+8]$$
$$= -17.$$

$$c_3 = -2 \sum_{\substack{\Delta v_i v_j v_k \\ e(v_i) \le e(v_j) \le e(v_k)}} e^2(v_k) e(v_j)$$
$$= -2\left[e^2(v_1)e(v_2) + e^2(v_4)e(v_3)\right]$$
$$= -2\left[2^2 \times 1 + 2^2 \times 1\right]$$
$$= -2[8] = -16.$$

We need to the following recursive formula (Newton's Identity) see [9], to prove the next theorem. Let B be an $n \times n$ matrix over the field of real numbers and let

$$P(\lambda) = det(\lambda I - B) = c_0\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + ... + c_{n-1}\lambda + c_n$$

be the characteristic polynomial of B, where I is a unite $n \times n$ matrix. Then

Theorem 3.4 Let $\lambda_1, \lambda_2, ..., \lambda_n$ be the maximum eccentricity eigenvalues of $M_e(G)$. Then

(1) $\sum_{i=1}^{n} \lambda_i = 0$.

(2) $\sum_{i=1}^{n} \lambda_i^2 = 2\sum_{i=1}^{n}(x_i + y_i)e^2(v_i)$,where

$x_i = |\{u \in N(v_i) : e(u) < e(v_i), 1 \le i \le n\}|$

and

$y_i = |\{u_j \in N(v_i), j > i : e(u_j) = e(v_i), 1 \le i \le n\}|$.

(3) $\sum_{i=1}^{n} \lambda_i^3 = 6\sum_{\substack{\Delta v_i v_j v_k \\ e(v_i) \le e(v_j) \le e(v_k)}} e^2(v_k)e(v_j)$.

Proof. The proof is consequences of Newton's identity and Theorem 3.1.

Corollary 3.5 *Let G be a self-centred graph with diameter $D(G)$ and let $\lambda_1, \lambda_2, ..., \lambda_n$ be the maximum eccentricity eigenvalues of G. Then* $\sum_{i=1}^{n} \lambda_i^2 = 2mD^2(G).$

Example 3.6 *Let $\lambda_1, \lambda_2, ..., \lambda_n$ be the maximum eccentricity eigenvalues of a graph G. Then*

(1) If $G = K_n$, then $\sum_{i=1}^{n} \lambda_i^2 = n(n-1)$.

(2) If $G = C_n$, then $\sum_{i=1}^{n} \lambda_i^2 = \begin{cases} n(n-1), & if\ n\ is\ odd; \\ n^2, & if\ n\ is\ even. \end{cases}$

(3) If $G = K_{r,s}$, then $\sum_{i=1}^{n} \lambda_i^2 = 8rs.$

Theorem 3.7 If G is a self-centred k-regular graph with diameter D, then kD is a maximum eccentricity eigenvalue of G and $EM_e(G) = DE(G)$.

Proof. Let G be a self-centered k-regular graph with diameter. Then $e(v) = D$, for every $v \in V(G)$. By the definition of $M_e(G)$, there are k entries equal to D in each raw of $det(\lambda I - M_e(G))$. Then by replacing the first raw of $det(\lambda I - M_e(G))$ by the sum of all raw we get that $(\lambda - kD)$ is a factor of $det(\lambda I - M_e(G))$. Thus, kD is a maximum eccentricity eigenvalue of G. Since, G is self-centred k-regular graph, it follows by the definitions of adjacency and maximum eccentricity matrices of G that $M_e(G) = DA(G)$ and hence if λ_i is an eigenvalue of G, then $D\lambda_i$, for every $1 \le i \le n$, is a maximum eccentricity eigenvalue of G, see [13]. Thus, $EM_e(G) = DE(G)$.

Theorem 3.8 If the maximum eccentricity energy of a graph G is rational, then it must be an even integer.

Proof. Let G be a graph of order n and let $\lambda_1, \lambda_2, ..., \lambda_n$ be the maximum eccentricity eigenvalues of G. Since, $\sum_{i=1}^{n} \lambda_i = 0$, then let $\lambda_1, \lambda_2, ..., \lambda_r$ be the positive eigenvalues of G and the remaining are non positive. Then

$$EM_e(G) = \lambda_1 + \lambda_2 + ... + \lambda_r - (\lambda_{r+1} + ... + \lambda_n)$$
$$= 2(\lambda_1 + \lambda_2 + ... + \lambda_r).$$

Since, $\lambda_1, \lambda_2, ..., \lambda_r$ are algebraic numbers, so is their sum and hence must be integer if $EM_e(G)$ is rational.

Theorem 3.9 *Let G be a graph of order n, size m and radius $r(G)$. If $\lambda_1(G)$ is the largest maximum eccentricity eigenvalue of G, then* $\lambda_1(G) \ge \frac{2mr(G)}{n}.$

Proof. Let G be a graph of order n and let λ_1 be the largest maximum eccentricity eigenvalue of G. Then from [3] we have $\lambda_1 \ge \max_{X \neq 0}\left\{\frac{X^tAX}{X^tX}\right\}$, where X is any nonzero vector and X^t is its transpose and A is a matrix . By set $X = J = (1,1,...,1)^t$. Then we have

$$\lambda_1 \ge \frac{J^tM_e(G)J}{J^tJ} = \frac{\sum_{j=1}^{n}(\sum_{i=1}^{n} e_{ij})}{n}.$$

Since, $e_{ij} = \max\{e(v_i), e(v_j)\}$, if $v_iv_j \in E(G)$ and 0, otherwise, it follows that

$$\lambda_1 \ge \frac{2}{n}\sum_{i<j}\max\{e(v_i), e(v_j)\} = \frac{2}{n}\sum_{i=1}^{n}(x_i + y_i)e(v_i).$$

Since, $e(v) \ge r(G)$, for every $v \in V(G)$ and by Remark 3.2,

$\sum_{i=1}^{n}(x_i + y_i) = m$, than

$$\lambda_1 \geq \frac{2mr(G)}{n}.$$

4. MAXIMUM ECCENTRICITY ENERGY OF SOME STANDARD GRAPHS

In this section, we investigate the exact values of the maximum eccentricity energies of some well-known graphs.

Theorem 4.1 *Let K_n be the complete graph with $n \geq 2$ vertices. Then*

$$EM_e(K_n) = 2n - 2.$$

Proof. Since the complete graph K_n is self-centered $(n-1)$-regular graph with diameter, it follows from Theorem 3.7, that $EM_e(K_n) = DE(G) = E(G) = 2n-2.$

Theorem 4.2 *Let $K_{1,n-1}$ be the star graph of order $n \geq 3$. Then*

$$EM_e(K_{1,n-1}) = 4\sqrt{n-1}.$$

Proof. Let $K_{1,n-1}$ be the star graph with vertex set $v_0, v_1, \ldots. v_{n-1}$, where v_0 is the central vertex. Then

$$M_e(K_{1,n-1}) = \begin{pmatrix} 0 & 2 & 2 & 2 \\ 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ & & & \end{pmatrix}_{n\times n}.$$

The characteristic polynomial of $M_e(K_{1,n-1})$ is

$$P(K_{1,n-1}, \lambda) = \begin{vmatrix} \lambda & -2 & -2 & -2 \\ -2 & \lambda & 0 & 0 \\ -2 & 0 & \lambda & 0 \\ -2 & 0 & 0 & \lambda \\ & & & \end{vmatrix}_{n\times n}$$

$$= \lambda^{n-2}(\lambda^2 - 4(n-1)).$$

Hence, the maximum eccentricity spectrum of $K_{1,n-1}$ is

$$M_e\, Sp(K_{1,n-1}) = \begin{pmatrix} 2\sqrt{n-1} & 0 & -2\sqrt{n-1} \\ 1 & n-2 & 1 \end{pmatrix}.$$

Therefore, the maximum eccentricity energy of $K_{1,n-1}$ is $EM_e(K_{1,n-1}) = 4\sqrt{n-1}$.

Corollary 4.3 *Each positive integer $4p$, $p \geq 2$ is the maximum eccentricity energy of a star graph.*

Proof. Since the maximum eccentricity spectrum of K_{1,p^2} is

$$Sp(K_{1,p^2}) = \begin{pmatrix} 2p & 0 & -2p \\ 1 & n-2 & 1 \end{pmatrix}$$

it follows that $EM_e(K_{1,p^2}) = 4p.$

Theorem 4.4 *Let C_n be the cycle graph with $n \geq 3$ vertices. Then the maximum eccentricity energy of C_n is*

$$EM_e(C_n) = \begin{cases} (n-1)\csc\frac{\pi}{2n}, & if\ n \equiv 1(\text{mod }) 2; \\ 2n\csc\frac{\pi}{n}, & if\ n \equiv 2(\text{mod }) 4; \\ 2n\cot\frac{\pi}{n}, & if\ n \equiv 0(\text{mod }) 4; \end{cases}$$

Proof. Since the cycle graph C_n is self-centered 2-regular graph with diameter $D = \lfloor\frac{n}{2}\rfloor$, it follows from Theorem 3.7, that

$$EM_e(K_n) = \lfloor\frac{n}{2}\rfloor E(G) = \begin{cases} (\frac{n-1}{2})\dfrac{2}{\sin\frac{\pi}{2n}}, & if\ n \equiv 1(\text{mod }) 2; \\ (\frac{n}{2})\dfrac{4}{\sin\frac{\pi}{n}}, & if\ n \equiv 2(\text{mod }) 4; \\ (\frac{n}{2})\dfrac{4\cos\frac{\pi}{n}}{\sin\frac{\pi}{2}}, & if\ n \equiv 0(\text{mod }) 4; \end{cases}$$

$$= \lfloor\frac{n}{2}\rfloor E(G) = \begin{cases} (n-1)\csc\frac{\pi}{2n}, & if\ n \equiv 1(\text{mod }) 2; \\ 2n\csc\frac{\pi}{n}, & if\ n \equiv 2(\text{mod }) 4; \\ 2n\cot\frac{\pi}{n}, & if\ n \equiv 0(\text{mod }) 4; \end{cases}$$

Theorem 4.5 Let $K_{r,r}$, for $r \geq 2$ be the complete bipartite graph of order $n = 2r$. Then the maximum eccentricity energy of $K_{r,r}$ is

$$EM_e(K_{r,r}) = 2n.$$

Proof. Let $K_{r,r}$, $r \geq 2$, be the complete bipartite graph with vertex set $v_1, v_2, ...v_r, u_1, u_2, ..., u_r$. Then

$$M_e(K_{r,r}) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 2 & 2 & \cdots & 2 \\ 0 & 0 & \cdots & 0 & 2 & 2 & \cdots & 2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 2 & 2 & \cdots & 2 \\ 2 & 2 & \cdots & 2 & 0 & 0 & \cdots & 0 \\ 2 & 2 & \cdots & 2 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 2 & 2 & \cdots & 2 & 0 & 0 & \cdots & 0 \end{pmatrix}_{2r \times 2r}$$

The characteristic polynomial of $M_e(K_{r,r})$ is

$$P(K_{r,r}, \lambda) = \begin{vmatrix} \lambda & 0 & \cdots & 0 & -2 & -2 & \cdots & -2 \\ 0 & \lambda & \cdots & 0 & -2 & -2 & \cdots & -2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda & -2 & -2 & \cdots & -2 \\ -2 & -2 & \cdots & -2 & \lambda & 0 & \cdots & 0 \\ -2 & -2 & \cdots & -2 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -2 & -2 & \cdots & -2 & 0 & 0 & \cdots & \lambda \end{vmatrix}_{2r \times 2r}$$

$$= \lambda^{2(r-1)}(\lambda - 2r)(\lambda + 2r).$$

Hence, the maximum eccentricity spectrum of $K_{r,r}$ is

$$M_e\ Sp(K_{r,r}) = \begin{pmatrix} 2r & 0 & -2r \\ 1 & 2r-2 & 1 \end{pmatrix}.$$

Therefore, the maximum eccentricity energy of $K_{r,r}$ is $EM_e(K_{r,r}) = 4r = 2n.$

Definition 4.6 [5] A Friendship graph F_r for $r \geq 2$, is the graph constructed by joining r copies of K_3 graph with a common vertex. F_r graph has $n = 2r + 1$ vertices and diameter $D(F_r) = 2$.

Theorem 4.7 Let F_r be a friendship graph. Then the maximum eccentricity energy of F_r is

$$EM_e(F_r) = (4r - 2) + 2\sqrt{8r+1}.$$

Proof. Let F_r, $r \geq 2$, be the friendship graph with vertex set $v_0, v_1, ...v_{2r}$. Then

$$M_e(F_r) = \begin{pmatrix} 0 & 2 & 2 & 2 & 2 & \cdots & 2 & 2 \\ 2 & 0 & 2 & 0 & 0 & \cdots & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 2 & 0 & 0 & 0 & 2 & \cdots & 0 & 0 \\ 2 & 0 & 0 & 2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 0 & 0 & 0 & 0 & \cdots & 0 & 2 \\ 2 & 0 & 0 & 0 & 0 & \cdots & 2 & 0 \end{pmatrix}_{(2r+1) \times (2r+1)}$$

The characteristic polynomial of $M_e(F_r)$ is

$$P(F_r, \lambda) = \begin{vmatrix} \lambda & -2 & -2 & -2 & -2 & \cdots & -2 & -2 \\ -2 & \lambda & -2 & 0 & 0 & \cdots & 0 & 0 \\ -2 & -2 & \lambda & 0 & 0 & \cdots & 0 & 0 \\ -2 & 0 & 0 & \lambda & -2 & \cdots & 0 & 0 \\ -2 & 0 & 0 & -2 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -2 & 0 & 0 & 0 & 0 & \cdots & \lambda & -2 \\ -2 & 0 & 0 & 0 & 0 & \cdots & -2 & \lambda \end{vmatrix}_{(2r+1) \times (2r+1)}$$

$$= (\lambda - 2)^{(r-1)}(\lambda + 2)^r(\lambda^2 - 2\lambda + 8r).$$

Hence, the maximum eccentricity spectrum of F_r is

$$M_e\ Sp(F_r) = \begin{pmatrix} 1+\sqrt{8r+1} & 2 & -2 & 1-\sqrt{8r+1} \\ 1 & r-1 & r & 1 \end{pmatrix}.$$

Therefore, the maximum eccentricity energy of F_r is

$$EM_e(F_r) = (4r - 2) + 2\sqrt{8r+1}.$$

Definition 4.8 The crown graph S_n^0 for $n \geq 3$, is the graph with vertex set $\{v_1, v_2, ..., v_r, u_1, ..., u_r\}$ and edge set $\{u_i v_i : 1 \leq i, j \leq r, i \neq j\}$. Therefore, $n = |V(S_r^0)| = 2r$ and S_r^0 coincides with the complete bipartite graph $K_{r,r}$ with the horizontal edges removed.

Theorem 4.9 Let S_r^0 be a crown graph. Then the maximum eccentricity energy of S_r^0 is

$$EM_e(S_r^0) = 6n - 12.$$

Proof. Let S_r^0, $r \geq 3$, be the crown graph with vertex set $\{v_0, v_1, ...v_r, u_1, .., u_r\}$. Then

$$M_e(S_r^0) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 0 \\ 0 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{2r\times 2r}$$

The characteristic polynomial of $M_e(S_r^0)$ is

$$P(S_r^0,\lambda) = \begin{vmatrix} \lambda & 0 & \cdots & 0 & 0 & -1 & \cdots & -1 \\ 0 & \lambda & \cdots & 0 & -1 & 0 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda & -1 & -1 & \cdots & 0 \\ 0 & -1 & \cdots & -1 & \lambda & 0 & \cdots & 0 \\ -1 & 0 & \cdots & -1 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & 0 & 0 & 0 & \cdots & \lambda \end{vmatrix}_{2rx2r}$$

$$= (\lambda-(3r-3))(\lambda+(3r-3))(\lambda-3)^{(r-1)}(\lambda+3)^{(n-1)}.$$

Hence, the maximum eccentricity spectrum of S_r^0 is

$$M_e\,Sp(S_r^0) = \begin{pmatrix} 3r-3 & 3 & -3 & -(3r-3) \\ 1 & r-1 & r-1 & 1 \end{pmatrix}.$$

Therefore, the maximum eccentricity energy of S_r^0 is

$$EM_e(S_r^0) = 12r-12 = 6n-12.$$

5. BOUNDS FOR MAXIMUM ECCENTRICITY ENERGY

In this section, we established upper and lower bounds for the maximum eccentricity energy of a graph. Similar to McClelland's bounds for graph energy, bounds for maximum eccentricity energy are given in the following two theorems.

Theorem 5.1 Let G be a connected graph of order $n \ge 2$ and size m and let $r(G)$ and $D(G)$ be the radius and the diameter of G respectively. Then

$$r(G)\sqrt{2m} \le EM_e(G) \le D(G)\sqrt{2nm}\,.$$

Proof. Consider the Cauchy-Schwartiz inequality

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \le \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right).$$

By choose $a_i = 1$ and $b_i = |\lambda_i|$ and by Theorem 3.4, we get

$$\left(EM_e(G)\right)^2 = \left(\sum_{i=1}^{n} |\lambda_i|\right)^2 \le \left(\sum_{i=1}^{n} 1\right)\left(\sum_{i=1}^{n} \lambda_i^2\right) \le n\left(2\sum_{i=1}^{n}(x_i+y_i)e^2(v_i)\right).$$

Since, $e(v) \le D(G)$, for every $v \in V(G)$ and by Remark 3.2, $\sum_{i=1}^{n}(x_i+y_i) = m$, it follows that

$$EM_e(G) \le D(G)\sqrt{2nm}.$$

Now, since $\left(\sum_{i=1}^{n}|\lambda_i|\right)^2 \ge \sum_{i=1}^{n}\lambda_i^2$, it follows that $\left(EM_e(G)\right)^2 \ge 2\sum_{i=1}^{n}(x_i+y_i)e^2(v_i)$. Since, $e(v) \ge r(G)$, for every $v \in V(G)$, then

$$EM_e(G) \ge r(G)\sqrt{2m}.$$

Theorem 5.2 Let G be a connected graph G of size m and radius $r(G)$. Then

$$EM_e(G) \ge 2m\left(r(G) + \sqrt[n]{det(M_e(G))}\right)^2.$$

Proof. Since

$$(EM_e(G))^2 = (\sum_{i=1}^{n}|\lambda_i|)^2 = (\sum_{i=1}^{n}|\lambda_i|)(\sum_{i=1}^{n}|\lambda_i|) = \sum_{i=1}^{n}|\lambda_i|^2 + 2\sum_{i<j}|\lambda_i||\lambda_j|.$$

Using the inequality between the arithmetic and geometric means, we get

$$\frac{1}{n(n-1)}\sum_{i\ne j}|\lambda_i||\lambda_j| \ge (\prod_{i\ne j}|\lambda_i||\lambda_j|)^{1/[n(n-1)]}.$$

Hence, by this and Theorem 3.4, we get

$$\left(EM_e(G)\right)^2 \ge \sum_{i=1}^{n}|\lambda_i|^2 + n(n-1)(\prod_{i\ne j}|\lambda_i||\lambda_j|)^{1/[n(n-1)]}$$

$$\ge \sum_{i=1}^{n}|\lambda_i|^2 + n(n-1)(\prod_{i=1}|\lambda_i|^{2(n-1)})^{1/[n(n-1)]}$$

$$= \sum_{i=1}^{n}|\lambda_i|^2 + n(n-1)|\prod_{i=1}\lambda_i|^{2/n}$$

$$= 2\sum_{i=1}^{n}(x_i + y_i)e^2(v_i) + n(n-1)|\prod_{i=1}\lambda_i|^{2/n}.$$

Since, $n(n-1) \geq 2m$ and $e(v) \geq r(G)$ for every connected graph, it follows by using Remark 3.2, that

$$EM_e(G) \geq 2mr^2(G) + 2m(det(M_e(G))^{2/n}$$
$$= 2m\left(r(G) + \sqrt[n]{det(M_e(G))}\right)^2.$$

Similar to Koolen and Moultonâ€™s [11], upper bound for the maximum eccentricity energy $EM_e(G)$ of a graph G is given in the following theorem.

Theorem 5.3 Let G be a connected graph of order $n \geq 2$, size m and radius $r(G)$. Then

$$EM_e(G) \leq \frac{2mr(G)}{n} + \frac{m}{n}\sqrt{2n^2D^2(G) - 4mr^2(G)}.$$

Proof. Consider the Cauchy-Schwartiz inequality

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq (\sum_{i=1}^{n} a_i^2)(\sum_{i=1}^{n} b_i^2).$$

By choosing $a_i = 1$ and $b_i = |\lambda_i|$, we have

$$\left(\sum_{i=2}^{n} |\lambda_i|\right)^2 \leq \left(\sum_{i=2}^{n} 1\right)\left(\sum_{i=2}^{n} \lambda_i^2\right).$$

Hence, by Theorem 3.4 and Remark 3.2, we have

$$\left(EM_e(G) - |\lambda_1|\right)^2 \leq (n-1)(2\sum_{i=1}^{n}(x_i + y_i)e^2(v_i) - \lambda_1^2).$$
$$\leq (n-1)(2mD^2(G) - \lambda_1^2).$$

Therefore, $EM_e(G) \leq \lambda_1 + \sqrt{(n-1)(2mD^2(G) - \lambda_1^2)}$.

Let $f(x) = x + \sqrt{(n-1)(2m + mo(G) - x^2)}$.

For decreasing function $f(x)$, $f'(x) \leq 0$ implies that, $1 - \frac{x(n-1)}{\sqrt{(n-1)(2mD^2(G) - x^2)}} \leq 0$, and hence $x \geq \sqrt{\frac{2mD^2(G)}{n}}$. Since, $2mD^2(G) \geq n$ and from Theorem 3.9, $\lambda_1 \geq \frac{2mr(G)}{n}$, it follows that

$$\sqrt{\frac{2mD^2(G)}{n}} \leq \frac{2mD^2(G)}{n} \leq \lambda_1.$$ Thus

$$f(\lambda_1) \leq f\left(\frac{2mD^2(G)}{n}\right).$$ That means,

$$EM_e(G) \leq f(\lambda_1) \leq f\left(\frac{2mD^2(G)}{n}\right).$$

Therefore, by using that $n-1 \leq m$, for every connected graph,

$$EM_e(G) \leq \lambda_1 + \sqrt{(n-1)(2mD^2(G) - \lambda_1^2)}$$
$$\leq \frac{2mr(G)}{n} + \sqrt{m(2mD^2(G) - (\frac{2mr(G)}{n})^2)}$$
$$\leq \frac{2mr(G)}{n} + \frac{m}{n}\sqrt{2n^2D^2(G) - 4mr^2(G)}.$$

6. Conclusion

In this paper, we introduced a new matrix of a connected graph G called maximum eccentricity matrix $M_e(G)$. It depends on the underling graph and eccentricity on its vertices. We obtained some coefficients of the characteristic polynomial of the maximum eccentricity matrix. It is shown that if G is a connected self-centered k-regular graph with diameter D, then $M_e(G)=DA(G)$, where A(G) is the adjacency matrix of G. We are interested in studying mathematical aspects of the maximum eccentricity energy $E_M(G)$ of a graph . It is shown that if G is a self-centered k-regular graph with diameter D, then kD is a maximum eccentricity eigenvalue of G. Upper and lower bounds for the maximum eccentricity energy of a graph are found. It is shown that if the maximum eccentricity energy is rational number then it must be an even integer number. It is possible that the maximum eccentricity energy that we are considering in this paper may have some applications in chemistry as well as in other areas.

Open Problems

1- Characterize all graphs G which $E_M(G)=E(G)$.
2- Characterize all graphs which are hyperenergetic and hypoenergrtic under the maximum eccentricity energy.
3- Find the maximum eccentricity energy for some other family graphs and for some operations on graphs.
4- Find others bounds for the maximum eccentricity energy.

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